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GENERATION OF BEAMS OF THREE-DIMENSIONAL PERIODIC INTERNAL WAVES BY SOURCES OF VARIOUS TYPES

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The energy and force characteristics of periodic internal wave beams in a viscous exponentially stratified fluid are analyzed. The exact solutions of linearized problems of generation obtained by integral transformations describe not only three-dimensional internal waves but also the associated boundary layers of two types. The solutions not containing empirical parameters are brought to a form that allows a direct comparison with experimental data for generators of various types (friction, piston, and combined) of rectangular or elliptic shape. The stress tensor and force components acting on the generator are given in quadratures. In the limiting cases, the solutions are uniformly transformed to the corresponding expressions for the problems in a two-dimensional formulation.

Key words: *stratified fluid, internal waves, analytical methods, exact solution, Stokes and internal boundary layers.*

Introduction. Internal waves, which play an important role in the dynamics of the ocean, atmosphere, and other stratified media, have been studied analytically [1], numerically, and experimentally under laboratory and natural conditions. In analytical calculations of wave fields, the real boundary conditions on the generators are simulated by sets of singular sources, whose properties are postulated [2] or borrowed from ideal fluid theory [3]. In an analysis of the perturbations induced in a fluid by a horizontal cylinder which performs rectilinear or torsional oscillations of small amplitude, the parameters of the wave beams and boundary layers are calculated separately [4, 5]. The results obtained, which are used to determine the regions of applicability of asymptotic approximations, are usually given in integral form [4, 5], which prevents their practical application.

Accounting for all roots of the dispersion equation allows one to simultaneously calculate the parameters of two-dimensional internal waves and associated boundary layers [6]. The results of such calculations agree with experimental data [7]. The approach proposed in [6] extends the classical Stokes method [8] to inhomogeneous media. The satisfaction of the exact boundary conditions allows one to calculate both the waves and the fine structure of the boundary layers resulting from linear oscillations of segments of a flat surface in arbitrary directions. The nonlinear problem of wave generation by the boundary layers on a disk performing torsional oscillations is considered in [9].

In the case of three-dimensional motion, the boundary layer on the oscillating part of the plane is more complex and includes an analog of the Stokes layer in a homogeneous fluid and a special inner boundary layer [10]. The method of constructing the solution [9] is fairly universal and suitable for calculating flows for more complex motions of the wave-generating surfaces used in experiments to increase the wave amplitude. The goal of the present study is to estimate the dynamic characteristics of beams of three-dimensional periodic internal waves generated by sources of various shapes.

1. Governing Equations and Boundary Conditions. We study steady-state cyclic motion in a viscous incompressible exponentially stratified fluid whose density is related to height by $\rho_0(z) = \rho_{00}\exp(-z/\Lambda)$ (Λ is the scale; the z axis is opposite to the acceleration of gravity $\boldsymbol{g}$). The buoyancy frequency $N = \sqrt{g/\Lambda}$, and the kinematic viscosity ν is constant. The source of the motion is a part of an included surface which performs linear

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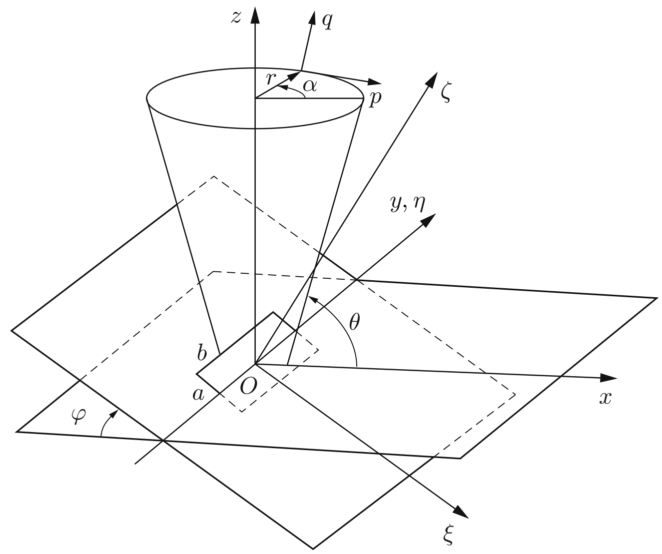

Fig. 1. Coordinate systems of the problem.

harmonic oscillations in a selected direction with a velocity magnitude u_0. The time dependence of all parameters is periodic, and the common coefficient $\exp(-i\omega t)$ is omitted below.

In the Boussinesq approximation, the linearized system of equations of motion is written as [10]

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\frac{\partial P}{\partial z} + \rho_0 \nu \Delta \boldsymbol{v} - \rho g \boldsymbol{e}_z, \qquad \frac{\partial \rho}{\partial t} - v_z \frac{\rho_0}{\Lambda} = 0, \qquad \operatorname{div} \boldsymbol{v} = 0, \tag{1}$$

where $\boldsymbol{v} = (v_x, v_y, v_z)$, ρ, and P are the velocity, density, and pressure perturbations, respectively, and $\boldsymbol{e}_z$ is the unit vector of the Oz axis. In the calculations, we use the standard conditions of the theory of internal waves: the weak stratification approximation $\Lambda \gg H$ (H is the maximum linear scale of the problem) and the condition of smallness of the viscosity $N\lambda_c^2 \gg \nu$ (λ_c is the characteristic length of the wave).

The form of the continuity equation for incompressible media allows us to introduce a toroidal–poloidal representation, in which the three velocity components $\boldsymbol{v} = (v_x, v_y, v_z)$ are expressed in terms of scalar functions Φ and Ψ [11] by the relation $\boldsymbol{v} = \nabla \times \boldsymbol{e}_z \Psi + \nabla \times (\nabla \times \boldsymbol{e}_z \Phi)$. Thus, system (1) becomes

$$(\omega^2 \Delta - N^2 \Delta_\perp - i\omega\nu\, \Delta^2)\Phi = 0, \qquad (\omega - i\nu\Delta)\Psi = 0, \tag{2}$$

where $\Delta = \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz}$ is the Laplacian and $\Delta_\perp = \partial^2_{xx} + \partial^2_{yy}$. In (2), the first independent equation is typical of internal wave theory and the second appears in descriptions of linear boundary layers in viscous fluids [2]. Following [12], the additional solutions due to the toroidal–poloidal representation and the factors corresponding to them are discarded in (2) from physical considerations.

The geometry of the problem and the coordinate systems used are shown in Fig. 1. The wave-generating surface is on an impenetrable plane which makes an angle φ with the horizon. In all cases, the coordinate origin is at the center of the oscillating region.

The gravity direction defines a laboratory coordinate system (x, y, z) attached to the fixed fluid. On the wave-generating surface there is the center of a local coordinate system (ξ, η, ζ) obtained by rotation of the system (x, y, z) through an angle φ around the y axis. In this case, the ξ and η axes are on the wave-generating surface and the ζ axis is normal to it. The wave cone is attached to two coordinate systems: a cylindrical system (r, α, z) and an associated system (q, p, α), in which the q axis makes an angle $\theta = \arcsin(\omega/N)$ with the horizon and is oriented in the wave propagation direction, and the p axis is normal to this direction (α is an angular variable):

$$\xi = x\cos\varphi + z\sin\varphi, \qquad \eta = y, \qquad \zeta = -x\sin\varphi + z\cos\varphi,$$

$$x = r\cos\alpha, \qquad y = r\sin\alpha, \qquad z = z,$$

$$p = r\sin\theta - z\cos\theta, \qquad q = r\cos\theta + z\sin\theta.$$

The boundary conditions for system (1) are the conditions of attachment on the entire separating plane $O\xi\eta$, including its moving and fixed parts, which for the scalar functions Φ and Ψ in (2) become

$$\begin{gathered}\cos\varphi\,\partial_\eta\Psi + [-\sin\varphi\,(\partial_\eta^2 + \partial_\xi^2) + \cos\varphi\,\partial_{\xi\zeta}^2]\Phi|_{\zeta=0} = u_\xi(\xi,\eta),\\ -(\cos\varphi\,\partial_\xi - \sin\varphi\,\partial_\eta)\Psi + \partial_\eta(\sin\varphi\,\partial_\xi + \cos\varphi\,\partial_\zeta)\Phi|_{\zeta=0} = u_\eta(\xi,\eta),\\ -\sin\varphi\,\partial_\eta\Psi + [-\cos\varphi\,(\partial_\eta^2 + \partial_\xi^2) + \sin\varphi\,\partial_{\xi\zeta}^2]\Phi|_{\zeta=0} = u_\zeta(\xi,\eta).\end{gathered} \tag{3}$$

At infinity, all perturbations damp. The unperturbed fluid is at rest.

2. General Solution of the Problem of Generation of Periodic Motion by the Oscillating Part of an Inclined Plane. The solution of system (2) is sought in the form of an expansion of the scalar functions Φ and Ψ in Fourier integrals:

$$\begin{gathered}\Phi = \int\limits_{-\infty}^{+\infty} \Big[A(k_\xi,k_\eta)\exp\left(ik_1(k_\xi,k_\eta)\zeta\right) + B(k_\xi,k_\eta)\exp\left(ik_2(k_\xi,k_\eta)\zeta\right)\Big]\exp\left(ik_\xi\xi + ik_\eta\eta\right)dk_\xi\,dk_\eta,\\ \Psi = \int\limits_{-\infty}^{+\infty} C(k_\xi,k_\eta)\exp\left(ik_3(k_\xi,k_\eta)\zeta + ik_\xi\xi + ik_\eta\eta\right)dk_\xi\,dk_\eta.\end{gathered} \tag{4}$$

The wave numbers k_1, k_2, and k_3 included in (4) are found by solving the dispersion equation, which in this case is written in multiplicative form

$$\Big\{\sin^2\theta\,(k_{1,2}^2 + k_\perp^2) - [(k_\xi\cos\varphi - k_{1,2}\sin\varphi)^2 + k_\eta^2] + i\sin\theta\,\delta_N^2(k_{1,2}^2 + k_\perp^2)^2\Big\}$$
$$\times\,[\delta_N(k_3^2 + k_\perp^2) - i\sin\theta] = 0. \tag{5}$$

Here $\delta_N = \sqrt{\nu/N}$ is the universal microscale, $\sin\theta = \omega/N$, and $k_\perp^2 = k_\xi^2 + k_\eta^2$. In addition to the frequency ratio characteristic of internal waves [1], the dispersion equation for the periodic motion in a viscous stratified fluid contains the universal microscale δ_N, which characterizes the boundary layers [6–9].

Equation (5) has three pairs of complex roots, one of which is regular in viscosity ($\operatorname{Im} k \sim \nu$), and the other two are singular ($\operatorname{Im} k \sim \nu^{-1/2}$). The regular roots describe a beam of conical internal waves, which also exists in an ideal fluid. In weakly stratified low-viscosity media, the roots of the dispersion equation (5) are found by methods of perturbation theory [13]. In the half-space $\zeta > 0$, the values of the roots are chosen such that the perturbations damp at infinity $\operatorname{Im} k_1 > 0$, $\operatorname{Im} k_2 > 0$, and $\operatorname{Im} k_3 > 0$.

A full classification of the regular and singular components of three-dimensional cyclic motion in viscous fluids taking into account compressibility, rotation, and stratification is given in [14]. In passing to a homogeneous fluid, the difference between the singular roots (5) vanishes, i.e., the boundary layers become identical and form a degenerate boundary layer.

The coefficients A, B, and C are determined by solving the system of linear equations resulting form substitution of solution (4) into boundary conditions (3) for the source velocity $\boldsymbol{u}(\xi,\eta)$, which are also written in spectral form $\boldsymbol{U}(k_\xi,k_\eta)$:

$$\boldsymbol{U} = \frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty} \boldsymbol{u}(\xi,\eta)\exp\left(-ik_\xi\xi - ik_\eta\eta\right)d\xi\,d\eta.$$

Of practical interest are several types of generators employed in laboratory experiments. Among these are the following generators: a friction generator (a rectangle or a disk performing linear oscillations its the surface), a piston generator (a rectangle or a disk oscillations normal to its surface), and a combined generator (two identical conjugate rectangles oscillating in antiphase along the normal to its surface). The values of the velocity and density perturbations in the wave beams and boundary layers on the generators are given in [10, 15]. The previously obtained expression used to calculate the perturbations of the pressure and forces acting on the wave-generating surface are not given in the present paper.

3. Calculation of the Forces Acting on the Wave-Generating Surface. The components of the forces acting on the unit surface of the generator ($\tilde{f}_i = -\sigma_{ik} n_k$) are defined by the stress tensor [16]

$$\sigma_{ik} = -P\delta_{ik} + \mu\Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\Big),$$

which is calculated using the previously calculated functions Φ and Ψ in the local coordinate system.

The total force acting on the wave-generating surface is given by the expression

$$F_i = \iint\limits_S \tilde{f}_i(x,y)\,dx\,dy = \sin\varphi \iint\limits_S f_i(\xi,\eta)\,d\xi\,d\zeta. \tag{6}$$

Following [10], the pressure and density distributions in the local coordinate system attached to the wave-generating surface are written as

$$P = -\rho_0\omega \int\limits_{-\infty}^{+\infty} \chi_1\Big(A\,\mathrm{e}^{ik_1\zeta} + B\,\mathrm{e}^{ik_2\zeta}\Big)\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta$$

$$-\,i\rho_0\nu \int\limits_{-\infty}^{+\infty} \chi_1\Big[A\,\mathrm{e}^{ik_1\zeta}(k_1^2 + k_\perp^2) + B\,\mathrm{e}^{ik_2\zeta}(k_2^2 + k_\perp^2)\Big]\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta,$$

$$\rho = \rho_0\Big[1 + \frac{i}{\omega\Lambda}\int\limits_{-\infty}^{+\infty}(\varepsilon_1^2 + k_\eta^2)(A\,\mathrm{e}^{ik_1\zeta} + B\,\mathrm{e}^{ik_2\zeta})\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta\Big],$$

where $\varepsilon_i = k_\xi\cos\varphi - k_i\sin\varphi$ and $\chi_i = k_\xi\sin\varphi + k_i\cos\varphi$.

For convenience of the further calculations, the stress tensor components, which in formula (6) are written in the laboratory coordinate system (x,y,z) are represented in the local coordinate system (ξ,η,ζ):

$$\sigma_{xx} = -P + 2\mu\,\frac{\partial v_x}{\partial x} = -P - 2\mu\int\limits_{-\infty}^{+\infty} Ck_\eta\varepsilon_3\,\mathrm{e}^{ik_3\zeta + ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta$$

$$-\,2\mu\int\limits_{-\infty}^{+\infty}\Big(A\,\mathrm{e}^{ik_1\zeta}\,\varepsilon_1^2\chi_1 + B\,\mathrm{e}^{ik_2\zeta}\,\varepsilon_2^2\chi_2\Big)\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta,$$

$$\sigma_{yy} = -P + 2\mu\,\frac{\partial v_y}{\partial y} = -P + 2\mu\int\limits_{-\infty}^{+\infty} Ck_\eta\varepsilon_3\,\mathrm{e}^{ik_3\zeta + ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta$$

$$-\,2\mu i\int\limits_{-\infty}^{+\infty} k_\eta^2\Big(A\,\mathrm{e}^{ik_1\zeta}\,\chi_1 + B\,\mathrm{e}^{ik_2\zeta}\,\chi_2\Big)\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta,$$

$$\sigma_{zz} = -P + 2\mu\,\frac{\partial v_z}{\partial z} = -P - 2i\mu\int\limits_{-\infty}^{+\infty}\Big[A\,\mathrm{e}^{ik_1\zeta}\,\chi_1(\varepsilon_1^2 + k_\eta^2) + B\,\mathrm{e}^{ik_2\zeta}\,\chi_2(\varepsilon_2^2 + k_\eta^2)\Big]\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta,$$

$$\sigma_{xy} = \mu\Big(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\Big) = \mu\int\limits_{-\infty}^{+\infty} C(\varepsilon_3^2 - k_\eta^2)\,\mathrm{e}^{ik_3\zeta + ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta$$

$$-\,i\mu\int\limits_{-\infty}^{+\infty} k_\eta\Big[A\,\mathrm{e}^{ik_1\zeta}\,\varepsilon_1\chi_1 + B\,\mathrm{e}^{ik_2\zeta}\,\varepsilon_2\chi_2\Big]\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta}\;dk_\xi\,dk_\eta,$$

$$\sigma_{xz} = \mu\Big(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\Big) = -\mu \int\limits_{-\infty}^{+\infty} Ck_\eta \chi_3 \, \mathrm{e}^{ik_3\zeta + ik_\xi \xi + ik_\eta \eta} \; dk_\xi \, dk_\eta$$

$$+ i\mu \int\limits_{-\infty}^{+\infty} \Big[A\,\mathrm{e}^{ik_1\zeta}(k_\eta^2\varepsilon_1 + \varepsilon_1^3 - \varepsilon_1\chi_1^2) + B\,\mathrm{e}^{ik_2\zeta}(k_\eta^2\varepsilon_2 + \varepsilon_2^3 - \varepsilon_3\chi_2^2)\Big]\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta} \; dk_\xi \, dk_\eta,$$

$$\sigma_{yz} = \mu\Big(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\Big) = \mu \int\limits_{-\infty}^{+\infty} Ck_\eta \chi_3 \, \mathrm{e}^{ik_3\zeta + ik_\xi \xi + ik_\eta \eta} \; dk_\xi \, dk_\eta$$

$$+ i\mu \int\limits_{-\infty}^{+\infty} (k_\eta^2\varepsilon_1 + \varepsilon_1^3 + k_\eta^2 + k_\eta\varepsilon_1^2)\Big(A\,\mathrm{e}^{ik_1\zeta} + B\,\mathrm{e}^{ik_2\zeta}\Big)\,\mathrm{e}^{ik_\xi\xi + ik_\eta\eta} \; dk_\xi \, dk_\eta.$$

The further calculations the viscous stress tensor component can be performed using numerical methods.

The expressions for the total forces taking into account damping in the boundary layers and energy transfer by the waves are defined by integrals of the viscous stress tensor component and are also not reduced to known functions. An objective of practical interest is to calculate the pressure and density perturbations, which are measured by independent methods.

4. Calculation of Pressure and Density Perturbations in the Fluid. The calculations performed are based on the results of asymptotic calculations [10] of the parameters of three-dimensional internal wave beams and two boundary layers. The perturbations produced by a friction generator of rectangular shape with sides a and b aligned with the ξ and η coordinate axes, which oscillates along the $O\xi$ axis at a velocity $\boldsymbol{u} = u_0\theta(a/2 - |\xi|)\theta(b/2 - |\eta|)\boldsymbol{e}_\xi$. Using the low viscosity approximation and assuming that the source size is smaller than the viscous wave scale $L_\nu = (\nu\Lambda/N)^{1/3}$ ($a \ll L_\nu$ and $b \ll L_\nu$), we calculate the pressure perturbations in the neighborhood of the source:

$$P_1^b \approx -i\rho_0\omega u_0\delta_\varphi \exp\Big(\frac{i-1}{\delta_\varphi} + \frac{i\pi}{4}\Big), \tag{7}$$

and at large distances from it ($q \gg a$ and $q \gg b$):

$$P_1^w \approx \frac{\rho_0\omega u_0 ab\delta_N}{2\pi^{3/2}\sqrt{|\mu|}}\,\sin\theta\,\Big[\cos\varphi\,\sin\theta\,\sin\Big(\frac{\pi}{4} - \alpha\Big) - \cos\theta\,\sin\varphi\Big]G\Big(\frac{1}{2}, p, q\Big). \tag{8}$$

The function $G(n, p, q)$ is given by the convolution of the solution for a point generator with an exponential function:

$$G(n, p, q) = \frac{1}{\sqrt{p\sin\theta + q\cos\theta}} \int\limits_0^{+\infty} dk_p \; k_p^n \exp\Big(ik_p p - \frac{k_p^3\delta_N^2 q}{2\cos\theta}\Big). \tag{9}$$

In formula (9), the integral is an analytical function of viscosity since the second term in the exponent vanishes uniformly as the viscosity tends to zero.

Similar expressions were analyzed previously in a study of internal waves in viscous fluids [1] (see also [2, formula (6.6); Appendix B]); however, the exponent n remained unknown in this case. Since in the modeling of the generator by multipoles, the properties of only the regular components of the solution are used, the calculated flow pattern cannot be matched to the real one, which contains boundary layers (and components that are singular in viscosity in the solutions of the equations of motion). The form of the function $G(n,\ p, q)$ suggests that the number n is not an integer, i.e., the solution of system (2) generally cannot be represented as an expansion in multipoles.

On the beam axis ($p = 0$) at large distances from the wave source, expression (9) is considerably simplified:

$$G(n, 0, q) = \frac{1}{3}\,\frac{1}{\sqrt{q\cos\theta}}\Big(\frac{2\cos\theta}{q\delta_N^2}\Big)^{(n+1)/3}\Gamma\Big(\frac{n+1}{3}\Big).$$

At $\alpha = -\pi/4$, the maximum pressure on the beam axis is

$$|P_1^w|_{\max} = \frac{\rho_0 \omega u_0 ab \sin\theta \, \sin(\theta - \varphi)}{6\sqrt{2}\, q\pi \sqrt{|\mu|}}.$$

Expressions (7) and (8) were calculated in a linear approximation, which is valid in the case of satisfaction of the inequalities $u_0 l_1/\nu \ll 1$ ($l_1 = ab/q$ is the ratio of the generator area to the distance from the source to the measurement point in the region of wave motions) and $u_0 \delta_N/\nu \ll 1$ for the boundary layers. The expressions given above has the meaning of the Reynolds number. The velocity scale u_0 is determined by the boundary conditions. Near the source, the length scale is specified by the universal microscale δ_N, and away from it, by the quantity l_1. The linear approximation is almost always valid in analyzing wave motions but can be violated near the wave-generating surface, where the shear boundary layers interact intensely with each other, generating waves of higher harmonics [16] and vortices.

If a part of the surface performs oscillations along the ζ axis normal to the plane of the source (a piston generator) at a velocity

$$\boldsymbol{u} = u_0 \theta(a/2 - |\xi|)\theta(b/2 - |\eta|)\boldsymbol{e}_\zeta,$$

the asymptotic expressions for the pressure in the wave beam become

$$P_2^w \approx -\frac{\rho_0 \omega u_0 ab}{\pi}\, \mathrm{e}^{-i\pi/4} \sqrt{\frac{\sin\theta}{\pi}}\, G\Big(-\frac{1}{2}, p, q\Big),$$

$$P_2^b \approx \frac{i+1}{\sqrt{|\mu|}}\, i\rho_0 \omega u_0 \delta_N \, \frac{\sqrt{\sin\theta}\, \cot\varphi}{\pi^2}\, W_2,$$

$$W_2 = \int\limits_{-\infty}^{+\infty} \frac{k_\eta^2 \cos\varphi + k_1^{(0)}\sigma}{k_\xi k_\eta}\, W_\delta(k_\xi, k_\eta)\, dk_\xi\, dk_\eta,$$

$$W_\delta(k_\xi, k_\eta) = \frac{\sin(k_\xi a/2)\sin(k_\eta b/2)}{k_\eta^2 \cos\varphi - k_\xi \sigma} \exp\Big[ik_\xi\Big(\xi - \frac{\sin\varphi\, \cos\varphi}{\mu}\,\zeta\Big) + ik_\eta \eta\Big],$$

$$\sigma = \Big(k_\xi \cos\varphi \sin^2\theta + \cos\theta \sin\varphi \sqrt{k_\xi^2 \sin^2\theta - k_\eta^2 \mu}\,\Big)/\mu,$$

$$\mu = \sin^2\varphi - \sin^2\theta, \qquad k_1^{(0)} = \Big(k_\xi \sin 2\varphi + 2\cos\theta \sqrt{k_\xi^2 \sin^2\theta - \mu k_\eta^2}\,\Big)/(2\mu).$$

The maximum pressure on the beam axis ($p = 0$) equals

$$|P_2^w|_{\max} = \frac{\rho_0 \omega u_0 ab}{3\pi\sqrt{\pi}} \Big(\frac{2\tan^2\theta\, \sin\theta}{q^4 \delta_N^2}\Big)^{1/6} \Gamma\Big(\frac{1}{6}\Big).$$

In this case, the structure of the inequality determining the condition of applicability of the linear approximation changes. For the wave component, it becomes $u_0 l_2/\nu \ll 1$. This condition includes the combination length scale $l_2 = ab/(q^{2/3}\delta_N^{1/3})$, which now contains four linear scales of the problem (the generator sizes, the distance to the observation point, and the universal microscale). The change in the structure of the inequality is due to the specificity of the wave generation mechanism, in particular, the higher effectiveness of the piston generator compared to the friction generator. The condition of linearization of the equations near the generator remains unchanged: $u_0 \delta_N/\nu \ll 1$.

For the longitudinal linear oscillations along the ξ axis performed by an ellipse with the a and b semi-axes aligned with the ξ and η axes, respectively, expression (6) becomes

$$U_\xi = \frac{u_0 ab}{2\pi\sqrt{k_\xi^2 a^2 + k_\eta^2 b^2}}\, J_1\Big(\sqrt{k_\xi^2 a^2 + k_\eta^2 b^2}\,\Big). \qquad (10)$$

In the particular case [the generator is a disk ($a = b = R$)], Eq. (10) implies that

$$U_\xi = \frac{u_0 R}{2\pi\sqrt{k_\xi^2 + k_\eta^2}}\, J_1\Big(R\sqrt{k_\xi^2 + k_\eta^2}\,\Big).$$

The expressions for the effectiveness of wave generation by the disk oscillating along the plane, and hence, and for the pressure distribution differ only slightly from (7) and (8):

$$P_3^w \approx \frac{\rho_0\omega\, u_0 R^2 \delta_N}{2\sqrt{\pi|\mu|}}\,\sin\theta\,\Big[\cos\varphi\,\sin\theta\,\sin\Big(\frac{\pi}{4} - \alpha\Big) - \cos\theta\,\sin\varphi\Big] G\Big(\frac{1}{2}, p, q\Big),$$

$$P_3^b \approx \rho_0\omega u_0 R\,\frac{1 - i}{\sqrt{|\mu|}}\,\sqrt{\sin\theta}\,\exp\Big(\frac{i-1}{\delta_\varphi}\Big)\begin{cases} 1/R, & \sqrt{\xi^2 + \eta^2} \leqslant R, \\ 0, & \sqrt{\xi^2 + \eta^2} > R. \end{cases}$$

It should be noted that in the approximation used for the fixed part of the generator plane ($\sqrt{\xi^2 + \eta^2} > R$) in the asymptotic expressions, the pressure vanishes.

For $\alpha = -\pi/4$, the maximum pressure on the beam axis is given by

$$|P_3^w|_{\max} = \frac{\rho_0\omega u_0 R^2 \sin\theta\,\sin(\theta - \varphi)}{6\sqrt{2}\,q\sqrt{|\mu|}}.$$

In this case, the condition of neglect of the convective term in the far field $u_0 l_3/\nu \ll 1$ includes the new characteristic length scale $l_3 = R^2/q$. The linearization condition near the source remains unchanged: $u_0\delta_N/\nu \ll 1$.

For the problem of wave generation by an ellipse with the a and b axis aligned with the ξ and η axes, respectively, which oscillates in the $O\zeta$ direction, the Fourier image of the source velocity distribution becomes

$$U_\zeta = \frac{ab}{2\pi\sqrt{k_\xi^2 a^2 + k_\eta^2 b^2}}\, J_1\Big(\sqrt{k_\xi^2 a^2 + k_\eta^2 b^2}\,\Big). \tag{11}$$

In the particular case of a disk generator ($a = b = R$), relation (11) implies that

$$U_\zeta = \frac{R}{2\pi\sqrt{k_\xi^2 + k_\eta^2}}\, J_1\Big(R\sqrt{k_\xi^2 + k_\eta^2}\,\Big),$$

$$P_4^w \approx -\frac{\rho_0\omega u_0 R^2}{\pi}\, \mathrm{e}^{-i\pi/4}\sqrt{\sin\theta}\, G\Big(-\frac{1}{2}, p, q\Big), \qquad P_4^b \approx \frac{i+1}{\sqrt{|\mu|}}\, i\rho_0\omega u_0\delta_N\,\frac{\sqrt{\sin\theta}\,\cot\varphi}{\pi^2}\, W_2.$$

The maximum pressure on the beam equals

$$|P_4^w|_{\max} = \frac{\rho_0\omega u_0 R^2}{3\sqrt{\pi}}\Big(\frac{2\tan^2\theta\,\sin\theta}{q^4\delta_N^2}\Big)^{1/6}\Gamma\Big(\frac{1}{6}\Big).$$

In this case, the inequality defining the linearization condition in the wave beam contains a new characteristic length scale $l_4 = R^2/(q^{2/3}\delta_N^{1/3})$, which differs from l_2. The linearization condition near the generator remains unchanged.

In practice, large-amplitude waves are produced using a more complex generator design whose parts move in different directions [17]. This generator is modeled by two identical rectangles conjugate on the side a, oscillates in antiphase along the ζ axis. The velocity of the generator is given by the expression

$$\boldsymbol{u} = u_0\theta(a/2 - |\xi|)[\theta(\eta)\theta(b/2 - \eta) - \theta(-\eta)\theta(b/2 + \eta)]\boldsymbol{e}_\zeta.$$

The compound source fits better to the structure of the wave field, produces larger wave perturbations in the near field, and is characterized by the following pressure distribution:

$$P_5^w \approx -\frac{\rho_0\omega u_0 ab}{2\pi^{3/2}}\, \mathrm{e}^{i\pi/4}\sin^{3/2}\theta\,\cos\Big(\frac{\pi}{4} - \alpha\Big) G\Big(\frac{1}{2}, p, q\Big),$$

$$P_5^b \approx -i\rho_0\omega u_0\delta_N\,\frac{\sqrt{\sin\theta}\,\cot\varphi}{\pi^2}\,\exp\Big(\frac{i-1}{\delta_\varphi}\Big) W_4,$$

TABLE 1

Wave Density Components
for Various Generators in the Low-Viscosity Approximation

Type of source	A_m^w	$f(\alpha,\theta,\varphi)$	n
Friction generator (rectangle)	$\frac{i}{4}\frac{u_0ab\delta_N}{\sqrt{\pi^3\lvert\mu\rvert}}$	f	3/2
Piston generator (rectangle)	$\frac{1-i}{(2\pi)^{3/2}}\frac{u_0ab}{\sqrt{\sin\theta}}$	1	1/2
Compound generator (rectangle)	$\frac{1-i}{(2\pi)^{3/2}}\frac{u_0ab^2\sqrt{\sin\theta}}{2}$	$\cos\left(\frac{\pi}{4}-\alpha\right)$	3/2
Friction generator (disk)	$\frac{i}{4}\frac{u_0R^2}{\sqrt{\pi\lvert\mu\rvert}}\,\delta_N$	f	3/2
Piston generator (disk)	$\frac{1-i}{(2\pi)^{3/2}}\frac{u_0\pi R^2}{\sqrt{2\sin\theta}}$	1	1/2

$$W_4 = \int\limits_{-\infty}^{+\infty} \frac{k_\eta^2 \sin\varphi + k_1^{(0)}\sigma}{k_\xi k_\eta}\, V_\delta\, dk_\xi\, dk_\eta,$$

$$V_\delta(k_\xi, k_\eta) = \frac{\sin(k_\xi a/2)\sin(k_\eta b/2)}{k_\eta^2\cos\varphi - k_\xi\sigma}\exp\Big[ik_\xi\xi + ik_\eta\eta + \frac{i-1}{4}\,\delta_\nu\zeta(k_\xi^2 + k_\eta^2)\Big].$$

For $\alpha = \pi/4$, the maximum pressure on the beam axis is given by the expression

$$|P_5^w|_{\max} = \frac{\sqrt{2}\,\rho_0\omega u_0 ab^2\sin^{3/2}\theta}{12q\delta_N^2}.$$

The linearization condition $u_0 l_5/\nu \ll 1$ for the wave beam includes the scale $l_5 = ab^2/(q\delta_N)$. Near the source, its shape remains former.

For all types of sources studied, the results of calculations of the density perturbation can be written in general form

$$\rho \approx \frac{2i\rho_0}{\omega\Lambda}\, A_m^w \sin^2\theta\, M_m(p,q), \qquad m = 1,\ldots,5,$$

$$M_m(n,p,q) = f(\alpha,\theta,\varphi)G(n,p,q), \qquad f = \cos\varphi\,\sin\theta - \sin(\pi/4-\alpha)\,\sin\varphi\,\cos\theta.$$

In the associated coordinate system (p,q), the expressions for the density perturbations in the wave differ from the expressions for the wave components of the velocities (see [18]) only in the coefficient $i\rho_0/(\omega\Lambda)$. The corresponding expressions given in Table 1 have a simple physical meaning. The coefficient A_m^w includes the source parameters (dimensions and position), the scales inducing the generation mechanism (δ_N for a friction generator), and functions of the angular position of the beam. These expression are inapplicable for small oscillation frequencies, for which methods of constructing nonuniform asymptotic relations do not work. The function f is defined by the angular characteristics of the problem, and the exponent n is the multipolarity index of the source.

In the low viscosity approximations, the viscous stress tensor components in wave beams at large distances from the source ($q \gg a$ and $q \gg b$) are written as

$$\sigma_{xx} = -p_m^w - 2i\rho_0\nu A_m^w\cos^2\alpha\,\sin 2\theta\,\cos\alpha\, Q_m(n,p,q),$$

$$\sigma_{yy} = -p_m^w + 4i\rho_0\nu A_m^w\cos\alpha\,\sin\alpha\,\sin\theta\,\cos^2\theta\, Q_m(n,p,q),$$

$$\sigma_{zz} = -p_m^w - 4i\rho_0\nu A_m^w \cos\theta\, \cos^2\theta\, Q_m(n,p,q),$$

$$Q_m(n,p,q) = f(\alpha,\theta,\varphi)G(n+1,p,q).$$

On the beam axis, the viscous stress tensor components are generally written as

$$\sigma^{(m)} \approx -\rho_0 A_m^w \frac{f(\alpha,\theta,\varphi)}{3\sqrt{q\cos\theta}}\Big(\frac{2\cos\theta}{q\delta_N^2}\Big)^{(n+1)/3}\Big[\omega\Gamma\Big(\frac{n+1}{3}\Big)\sin\theta - \nu\Big(\frac{2\cos\theta}{q\delta_N^2}\Big)^{1/3}\Gamma\Big(\frac{n+2}{3}\Big)\Big],$$

where the subscript $m = 1$ corresponds to a friction generator (rectangle), $m = 2$ to a piston generator (rectangle), $m = 3$ to a compound generator (rectangle), $m = 4$ to a friction generator (disk), and $m = 5$ to a piston generator (disk). The values of the parameters A_m^w and $f(\alpha,\theta,\varphi)$ in the above formula for the different types of generator are listed in Table 1. In this case, the multipolarity index n takes the following values: $n = 5/2$ for friction sources (rectangle and disk) and compound sources (rectangle) and $n = 3/2$ for piston sources (rectangle and disk).

An analysis of the expressions shows that the pressure, density, and the viscous stress tensor components are proportional to the area of the wave source and are determined by the geometry of the problem [coefficient $f(\alpha,\theta,\varphi)$], the distance from the source q, and the universal microscale δ_N^β, where the parameter β depends on the type of source.

Table 1 gives the values of the coefficients A_m^w, $f(\alpha,\theta,\varphi)$, and the multipolarity index n in the expressions for the vertical components of the force acting on the generator:

$$F_z \approx i\rho_0\omega \sin\theta\, \sin^2\varphi\, \sin\alpha\, f(\alpha,\theta,\varphi)A_m^w \int\limits_{-a/2}^{a/2} dp \int\limits_{-b/2}^{b/2} dq\, G(n+1,p,q), \qquad m = 1,\ldots,5.$$

The expressions obtained above are unsuitable for low oscillation frequencies, for which the employed asymptotic computational method is inapplicable.

Conclusions. A procedure is developed to solve the completely linearized problem of wave generation by compact sources of three-dimensional internal waves in a viscous exponentially stratified fluid. The solution obtained is full, satisfies the exact boundary conditions, and describes the structure of the wave cone and the periodic boundary layers of two types.

The waves and boundary layers form a unified system of periodic motions in a continuously stratified fluid. Despite the difference between the scales, the waves and boundary layers are formed and disappear simultaneously. The properties of the wave beam depend on the shape and direction of the generator oscillation line. The fields of the density, pressure, viscous stress tensor, and the vertical force component were calculated for several (friction, piston, and compound) types of generator of rectangular or elliptic shapes and are given in the form of quadratures, which are convenient for analytical and numerical estimations. A comparative analysis of the effectiveness of different generators is conducted in [17].

For some cases, where by virtue of the symmetry of the problem (the generator in the shape of a horizontal disk), the expressions for the vertical force component are simplified, the structure of the wave beams is visualized in [18].

A more detailed consideration of molecular effects leads to inclusion of the equations of conservation of thermal energy and mass in system (1). In this case, the flow field has special boundary layers induced by diffusion on the roughness of the bounding surface [19] and additional pairs of diffusion and temperature boundary layers [20], whose influence should also be taken into account in analyzing fluid dynamics.

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